COLO-HEP-375
hep-th/9609211
September 1996

Anomaly cancellation in M-theory

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Abstract

We show the complete cancellation of gauge and gravitational anomalies in the M-theory of Horava and Witten using their boundary contribution, and a term coming from the existence of two and five-branes. A factor of three discrepancy noted in an earlier work is resolved. We end with a comment on flux quantization.

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In the M-theory of Horava and Witten [1] several consistency checks associated with anomaly cancellation were verified. However one test which involved a numerical coefficient of a purely gravitational anomaly was not carried out. In an earlier paper by the author [2][1] the coefficient of a certain M-theory Green-Schwarz term [3] was determined, but there appeared to be a factor of three discrepancy with the expression (3.12) of [1]. In this short note we review the anomaly cancellation argument and find that the M-theory topological terms do indeed cancel both gauge and gravitational anomalies. Finally we comment on flux quantization in M-theory in the light of a recent paper by E. Witten[4].

We work with the "downstairs" version of the theory, i.e. on an 11-D manifold $M = M_{10} \times S^1/Z_2$. The topological term in the low energy effective action of M-theory is

$$-\frac{1}{\kappa^2}\frac{1}{6}\int_M C \wedge K \wedge K. \tag{0.1}$$

In the above[2] C is the three form gauge field of 11D supergravity and $K = dC$.[3] In the Horava-Witten theory the the manifold M has a boundary which consists of two disconnected components on each of which E_8 gauge fields live, so that on dimensional reduction to ten dimensions one gets the low energy effective action of the Heterotic $E_8 \times E_8$ theory. On checking the supersymmetry of the resulting theory it was found by

[1] In this paper it was also found that the dimensionless ratio of gauge and gravitational couplings determined by anomaly cancellation in [1] (see below) checks with a result obtained from string dualtiy and D-brane methods, thus giving an additional test of M-theory.

[2] It should be noted that (0.1) has a factor two compared to the usual term since we are working in the "downstairs" version of the theory with $M = M_{10} \times S^1/Z_2$ where the integral goes over half the volume of the "upstairs'" version where $M = M_{10} \times S^1$ and the fields are Z_2 symmetric.

[3] Our 11 D supergravity conventions and definitions are the same as in [3]. In particular we define a p-form gauge field as $A = \frac{1}{p!}A_{I_1\ldots I_p}dx^{I_1}\ldots dx^{I_p}$. The field strength is then $F = dA$ or in components $F_{I_1\ldots I_{p+1}} = (p+1)\partial_{[I_1}A_{I_2\ldots I_{p+1}]}$ with unit strength anti-symmetrization. The comparison with the notation of Horava and Witten is as follows. $K = \sqrt{2}G$, $C = \sqrt{2}C^{HW}$ where $C^{HW} = C^{HW}_{IJK}dx^I dx^J dx^K, G^{HW} = dC^{HW} = \frac{1}{4!}G_{IJKL}dx^I dx^J dx^K dx^L$ are the three form gauge field and field strength (called C and G in [1]) as defined by Horava and Witten. Our indices I,J,K,L, run from 1 to 11 whilst indices A,B,C,D run from 1 to 10.

Horava and Witten[4] that one needed to have, on each component of the boundary,

$$K|_{\partial M} = \frac{\kappa^2}{2\lambda^2}\hat{I}_4. \tag{0.2}$$

In the above λ is the gauge field coupling,

$$\hat{I}_4 = \frac{1}{2}\mathrm{tr}\mathcal{R}^2 - \mathrm{tr}F^2, \tag{0.3}$$

F is an E_8 gauge field strength, and $\mathcal{R}$ is the curvature two form. Now defining $Q_3 = \frac{1}{2}\omega_{3L} - \omega_3$, where the two omegas correspond to the Lorentz and gauge Chern-Simons forms, we have the standard descent equations,[5]

$$\hat{I}_4 = dQ_3, \quad \delta Q_3 = dQ_2^1. \tag{0.4}$$

where δ is a gauge and local Lorentz variation and Q_2^1 is a two-form that is linear in the gauge parameter.

From (0.2), the relation $K = dC$, and the first equation in (0.4), we have (up to an irrelevant exact form)

$$C|_{\partial M} = \frac{\kappa^2}{2\lambda^2}Q_3. \tag{0.5}$$

Hence from the second equation in (0.4)

$$\delta C|_{\partial M} = \frac{\kappa^2}{2\lambda^2}dQ_2^1. \tag{0.6}$$

Now clearly we may extend this variation to the bulk by writing

$$\delta C = d\Lambda, \quad \Lambda|_{\partial M} = \frac{\kappa^2}{2\lambda^2}Q_2^1. \tag{0.7}$$

Hence we have from (0.1), and (0.7)

$$\begin{aligned} \delta W &= -\frac{1}{\kappa^2}\frac{1}{6}\int_M d\Lambda \wedge K \wedge K \\ &= -\frac{1}{\kappa^2}\frac{1}{6}\left(\frac{\kappa^2}{\lambda^2}\right)^3 \frac{1}{2}\int_{\partial M} Q_2^1 \wedge \frac{\hat{I}_4^2}{4}, \end{aligned} \tag{0.8}$$

[4]See equation (2.20) of the second paper of [1].

[5]See for example reference [6], chapter 13.

where to get the second equality we have used Stokes' theorem, $dK = 0$, and (0.2).

Now the boundary theory is anomalous and the variation of the quantum effective action Γ is given by[6].

$$\delta\Gamma = -\frac{1}{48(2\pi)^5}\int_{\partial M} Q_2^1 \wedge \left(-\frac{\hat{I}_4^2}{4} + X_8\right), \tag{0.9}$$

where $X_8 = -\frac{1}{8}\mathrm{tr}\mathcal{R}^4 + \frac{1}{32}(\mathrm{tr}\mathcal{R}^2)^2$. Cancellation of the $\hat{I}_4^2$ part of the anomaly then determines

$$\eta^{-1} \equiv \frac{\kappa^4}{\lambda^6} = \frac{1}{4(2\pi)^5} \tag{0.10}$$

as in [1].

Now as shown in [3] the existence of two and five branes in the theory implies that there is an additional topological (Green-Schwarz) term in M-theory.[7] This is given by,

$$W_5 = \left(\frac{(2\pi)^2}{2\kappa^2}\right)^{1/3} \frac{1}{24(2\pi)^4} 2\int_M C \wedge X_8. \tag{0.11}$$

The first factor in the equation above was obtained from the relation $T_2 = \left[\frac{(2\pi)^2}{2\kappa^2}\right]^{1/3}$, which was originally determined using D-brane methods [2], but after correcting a factor of two in the quantization formula of [3] as discussed in the appendix to [2], it can also be fixed purely within M-theory.[8]

Using equations (0.7), Stokes' theorem and $dX_8 = 0$ we have

$$\begin{aligned} \delta W_5 &= \left(\frac{(2\pi)^2}{2\kappa^2}\right)^{1/3} \frac{1}{24(2\pi)^4} 2\int_M d\Lambda \wedge X_8 \\ &= \frac{1}{48(2\pi)^5}\int_{\partial M} Q_2^1 \wedge X_8. \end{aligned} \tag{0.12}$$

[6]The numerical coefficient in (0.9) is fixed by standard methods. See for example [6] equation (13.3.41), (13.4.5) the line before (13.5.6) and equations (13.5.5) and (13.5.8). The form of the anomaly is given in [1].

[7]The existence of this term may also be inferred from an earlier string theory calculation [5].

[8]There is an extra factor of 2 compared to equation (0.20) of [2] because we are in the "downstairs" theory - see footnote 2.

In the last equation we have used the value of η given in (0.10). Thus we have the complete cancellation of the anomalies in the Horava-Witten M-theory,

$$\delta W + \delta W_5 + \delta\Gamma = 0. \tag{0.13}$$

While the above was being written up, a paper by E. Witten appeared [4] in which, *inter alia*, some issues of normalization in M-theory were discussed using index theory. To conclude this note we would like to make some related comments. Equation (0.2) may be rewritten (after using 0.10) as

$$\frac{G}{2\pi}|_{\partial M} \equiv \left[\frac{(2\pi)^2}{2\kappa^2}\right]^{1/3} \frac{K}{2\pi}|_{\partial M} = w(V) - \frac{\lambda}{2}, \tag{0.14}$$

where $w = F \wedge F/(16\pi^2)$ has integer valued periods, being the second Chern class of the E_8 bundle, and $\lambda \equiv p_1/2 = \mathcal{R} \wedge \mathcal{R}/(16\pi^2)$ which is half the Pontryagin class p_1 of the tangent bundle, also has integer-valued periods for a spin manifold so that in general as pointed out in [4], $G/(2\pi)$ has half integer periods . By considering a 4-cycle (C) in the bulk that is homologous to one in the boundary (C') this result was extended in [4] to the statement

$$\int_C \left(\frac{G}{2\pi} - \frac{\lambda}{2}\right) \ \epsilon\ \mathcal{Z} \tag{0.15}$$

There is an alternate way to get the normalization of $G/(2\pi)$. This follows from the fact that $K = dC$ and C is the three form field coupling to the 2-brane of M-theory. In earlier work this was given as $T_2 \int_C K/2\pi \ \epsilon\ \mathcal{Z}$[9] but due to anomalies of fermionic determinants in odd dimensions, it was pointed out in [4] that this gets modified to

$$\int_C \left(\frac{T_2 K}{2\pi} - \frac{\lambda}{2}\right) \ \epsilon\ \mathcal{Z} \tag{0.16}$$

The consistency of the two normalizations is then a consequence of the relation $T_2 = \left[\frac{(2\pi)^2}{2\kappa^2}\right]^{1/3}$ derived in [2]. Note that this is a check on the consistency of M-theory that is

[9]see for example [3] and the appendix of [2].

independent of the check coming from the pure gravity anomaly cancellation.[10]

Acknowledgments

I wish to thank E. Witten for the suggestion to check M-theory consistency conditions, and S. Chaudhuri and P. Horava for discussions. This work is partially supported by the Department of Energy contract No. DE-FG02-91-ER-40672.

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[10] As noted in [2] (equation (8), (11) and the discussion after the latter equation), the calculation of T_2 can be done in a fashion that is completely independent of M-theory quantization conditions.